Reproducing Calculations for the Analytical Hierarchy Process

Booz | Allen | Hamilton
International Infrastructure Team

Introduction

Booz Allen supports clients in the application of the Analytical Hierarchy Process (AHP) to decision support. We use commercial software produced by the Expert Choice Corporation. This software accepts input from the clients Subject Matter Experts and produces the calculations immediately for use in the elicitation process. We can implement the AHP in an ongoing dialog with the client that continues without the interruptions that would be required to develop the models using more basic software. However, clients often ask for independent verification of the calculations so that the use of Expert Choice software is not used as a black box of unknown quality.

Therefore, we set out to produce the calculations needed to perform the analysis for the AHP) independent of the Expert Choice software. For quality assurance purposes, we wished to verify the results for specific exercises calculated by Expert Choice. For this we used a combination of two software: Matlab and MicroSoft Excel.

Mathematical Packages

Matlab, an abbreviation for *matrix laboratory*, is a high-performance language for technical computing produced by MathWorks, Inc., Natick MA. It integrates computation, visualization, and programming in a user-friendly environment where problems and solutions are expressed in a familiar mathematical notation. Typical uses for Mathlab are math and computation; alogorithm development; modeling, simulation, and prototyping; data analysis, exploration, and visualization; scientific and engineering graphics; application development, including graphical user interface building. We chose Matlab because it employs matrix algebra and can perform the calculations we need for AHP such as raising a matrix to any specified arbitrary power.

AHP Calculations

The Analytical Hierarchy Process as stated by Thomas Saaty(1), allows one to achieve a powerful economy of thought by bring all relevant factors into a hierarchial decomposition system, with the objectives and functions represented in the higher levels and structure represented in lower levels. The AHP treats different components of a system and their interactions as a whole and allows feedback. One uses judgements or other data to make careful quantified tradeoffs among relevant criteria. The Expert Choice software that we used to collect data and calculate results incorporates the steps outlined by Thomas Saaty(1). We used data and calculations obtained from expert elicitation on the chemical treatment of spent nuclear fuel to obtain fissile nuclear material.

To verify the results obtained from the Expert Choice software, we used Matlab and Excel to calculate the exact solution. The most demanding mathematical task was to take the reciprocal matrix and raise them to arbitrarily large powers and dividing the sum of each row by the sum of the elements to obtain the eigenvalues. With Matlab we were able to raise each individual matrix to an arbitrary power; we chose to start with the power of five. We raised the matrices to higher powers until we got the desired accuracy in the eigenvalues. The breaking point to which all the individual matrices started to produce the required accuracy of three places to the right of the decimal point was at the power of ten or less. Just to double check we continued raising the matrices to higher powers until we got to the power of fifteen. The resulting eigenvalues continued to converge with increasing accuracy. Next we used Excel to sum each row and all the individual numbers in each matrix. Then we divided the sum of each row by the total sum of the elements of each matrix and tabulated the results. Our independently calculated values were the same as those produced by Expert Choice. Reference 1 also gave other methods of obtaining the solution, but this only produced approximations with no ability to produce results with increasing accuracy. Although we used two different software programs to perform these calculations, with a little more set up of the required algorithms, all the calculations should be able to be done on Matlab.

Calculation of Specific AHP Parameters

Aggregated Weights from Individual SME Pair-wise Comparisons. To get the aggregated weights from the individual pair-wise comparisons, we calculate the geometric mean of the SME input to each pair-wise comparison. We do this by multiplying the numerical values of the SME's inputs together and taking the *m*th root of the product where *m* is the number of SMEs that are providing input. We construct a reciprocal matrix using the aggregated pair-wise comparisons and determine the weightings as the eigenvalues of the matrix.

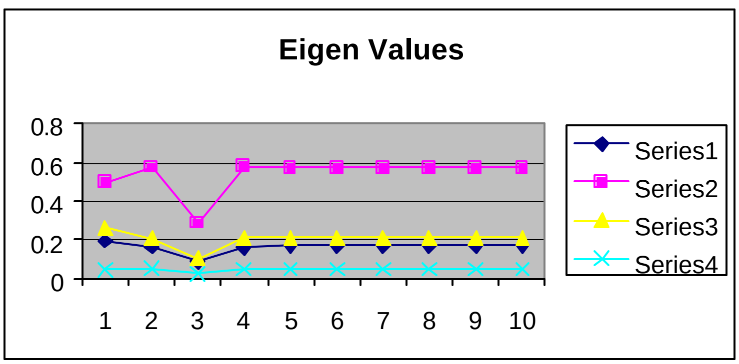
We can approximate the eigenvalues of the matrix by hand calculation to the precision desired for use in this verification process. However, the calculation is considerably expedited using Matlab software to conduct the matrix algebra. In either case, we raise the matrix to an arbitrary power, and normalize the sum of the rows to approximate the eigenvalues. Accuracy increases as we increase the power to which we raise the matrix. The power that we require for a specified accuracy relates to the inconsistency in the reciprocal matrix introduced by the SMEs.

Pairwise Comparisons		Mech Ops	Vs	D & Cond
SME #1		0.142857		7
SME #2		0.142857		7
SME #3		0.333333		3
SME #4		0.142857		7
SME #5		0.2		5
SME #6		6		0.166667
SME #7		0.5		2
SME #8		0.142857		7
Geometric Mean		0.309086		3.235343
Recipricol of GM		3.235343		0.309086

The aggregated weight of each pair-wise comparison is the geometric mean of the individual comparisons.

Combination Matrix			Mech Ops	D& Cond	Sep	Conv
Mech Ops			1.0000	0.3091	0.7267	4.1683
D& Cond			3.2353	1.0000	4.1955	7.3750
Sep			1.3761	0.2384	1.0000	5.7736
Conv			0.2399	0.1356	0.1732	1.0000
matrix^5					***Relative Weights***	
					mat result	**ec results**
0.286	0.0929	0.2489	1.0573	1.6851	0.170334	0.170
0.9595	0.3114	0.8357	3.5456	5.6522	0.571339	0.571
0.351	0.1139	0.3054	1.2977	2.068	0.209039	0.209
0.0828	0.0269	0.072	0.3059	0.4876	0.049288	0.049
9.8929						

The reciprocal matrix of the aggregated pair-wise comparisons yields the relative weights for each criterion.



For this matrix, eigenvalues are approximated to greater than three decimal point accuracy with the reciprocal matrix raised to a power greater than four. Each series represents one of four different criterion, Mech ops, D & Con, Sep, and Conv.

Individual SME Scores for Fuel Groups. To calculate the individual SME weighting of criteria, we form the reciprocal matrix based on the pair-wise comparisons and determine the individual SME weightings as the eigenvalues of the matrix. From the individual SME weighting, we calculate the total score afforded to each fuel group by summing the products, “rating for the fuel group in each criterion” times the “rating for that criterion.”

			Mech Ops	D & Cond	Sep	Conv
SME #2 weights on criteria			0.148	0.572	0.234	0.047
SME #2 on specific fuel group			0.063	0.002	0.011	0.091
Products			0.009324	0.001144	0.002574	0.004277
Sum						0.017319
(to 3 decimal places)						0.017

The weighting of each fuel group by an individual SME is the sum of product of relative weight times the fuel rating for each criterion.

Combined Score for Each Fuel Group. We combine the individual scores for each fuel group by taking the arithmetic average of SME scores for a criteria and multiplying by the aggregated weight for that criteria. We multiply that average by the aggregated weight for that criterion. Then we sum the four products (one for each criterion) to produce the combined score for each fuel group.

SPR w/o	Mech Ops	D & Cond	Sep	Conv
SME #1	0.063	0.002	0.011	0.091
SME #2	0.125	0.002	0.011	0.091
SME #3	0.063	0.002	0.011	0.091
SME #4	0.063	0.002	0.011	0.091
SME #5	0.125	0.002	0.011	0.091
SME #6	0.063	0.002	0.011	0.091
SME #7	0.125	0.002	0.011	0.091
SME #8	0.063	0.002	0.011	0.091
SUM	0.69	0.016	0.088	0.728
AVG	0.08625	0.002	0.011	0.091
Aggregated	0.17	0.571	0.209	0.049
Product	0.014663	0.001142	0.002299	0.004459
Sum				0.022563
(to 3 decimal places)				0.023

The aggregated weight for each fuel group is determined by combining the individual scores.

Separations Workshop Results:

Individual results:

Expert #1		***Expert #2***		***Expert #3***		***Expert #4***	
mat	***EC***	***mat***	***EC***	***mat***	***EC***	***mat***	***EC***
0.1443	0.144	0.0700	0.070	0.1479	0.148	0.1046	0.105
0.6221	0.622	0.5383	0.538	0.5718	0.572	0.6533	0.653
0.1999	0.200	0.3215	0.322	0.2335	0.234	0.1670	0.167
0.0335	0.033	0.0700	0.070	0.0466	0.047	0.0748	0.075

Expert #5		***Expert #6***		***Expert #7***		***Expert #8***	
mat	***EC***	***mat***	***EC***	***mat***	***EC***	***mat***	***EC***
0.1662	0.166	0.6346	0.635	0.2522	0.252	0.1172	0.117
0.5941	0.594	0.2451	0.245	0.5164	0.516	0.6117	0.612
0.1945	0.195	0.0906	0.091	0.1937	0.194	0.2197	0.22
0.0450	0.045	0.0296	0.030	0.0375	0.038	0.0512	0.051

We calculated the results to four decimal point accuracy and rounded to three decimal places to verify results of the Expert Choice Software program.

Combination results:

Combo	
mat	***EC***
0.1703	0.170
0.5713	0.571
0.2089	0.209
0.0492	0.049

Conclusions:

The matrix calculations with Matlab verify the results computed by Expert Choice for the SME input to the separations workshop. Matlab gave us a simpler and more accurate way of calculating the Expert Choice number then doing them by hand. The hand calculations could introduce operator error whereas Matlab gave us the power to do a multiple amount of functions at one time with a greater opportunity to reduce human error.

For additional info, contact LaKeisha McFarland or Martin Seitz of the Nuclear and Nonproliferation Group, ITT at (202) 626-1050.

References

1 Saaty, T. L., The Analytical Hierarchy Process, New York: McGraw-Hill, 1980.